

PRODUCT-FREE SETS WITH HIGH DENSITY

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Dedicated to Professor Andrzej Schinzel on his 75th birthday

ABSTRACT. We show that there are sets of integers with asymptotic density arbitrarily close to 1 in which there is no solution to the equation $ab = c$, with a, b, c in the set. We also consider some natural generalizations, as well as a specific numerical example of a product-free set of integers with asymptotic density greater than $1/2$.

1. INTRODUCTION

We say a set of integers \mathcal{S} is *product-free* if whenever $a, b, c \in \mathcal{S}$ we have $ab \neq c$. Similarly, if $\mathcal{S} \subset \mathbb{Z}/n\mathbb{Z}$, we say \mathcal{S} is product-free if $ab \not\equiv c \pmod{n}$, whenever $a, b, c \in \mathcal{S}$. Clearly, if \mathcal{S} is a product-free subset of $\mathbb{Z}/n\mathbb{Z}$, then the set of integers congruent modulo n to some member of \mathcal{S} is a product-free set of integers. For a positive integer n , let $D(n)$ denote the maximum value of $|\mathcal{S}|/n$ where \mathcal{S} runs over all product-free subsets of $\mathbb{Z}/n\mathbb{Z}$. (Here $|\mathcal{S}|$ denotes the cardinality of a set \mathcal{S} .)

In a recent paper, the third author and Schinzel [9] obtained an upper bound on $D(n)$ valid for a large set of n . They showed that $D(n) < 1/2$ whenever n is not divisible by a square with at least 6 distinct prime factors. Further, those numbers which are divisible by a square with at least 6 distinct prime factors form a set of asymptotic density about 1.56×10^{-8} . Originally they suspected that $D(n) < 1/2$ might hold for all n .

In this paper we show that for each real number $\epsilon > 0$ there is some number n with $D(n) > 1 - \epsilon$. Thus, there are product-free sets of integers with asymptotic density arbitrarily close to 1. Stated this way, the result is best possible, since no product-free set can have density 1. Indeed, if \mathcal{S} is a product free set of positive integers and a is the least member of \mathcal{S} , then it is easy to see that the upper density of \mathcal{S} is at most $1 - 1/(2a)$; see Remark 2.7.

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A consequence of our main result is that the set of integers n having $D(n) > 1 - \epsilon$ has a positive lower density. This follows using the property that $D(mn) \geq D(n)$ for all positive integers m, n . If $D(n_0) > 1 - \epsilon$, then it shows that $D(n) > 1 - \epsilon$ holds for every multiple of n_0 , and so it holds for a set of positive integers n of positive lower density. Furthermore the set $\mathcal{N}(u) = \{n \geq 1 : D(n) > u\}$ has a well-defined logarithmic density $\delta(u)$ which is positive for $0 < u < 1$. In Theorem 2.1 we obtain a quantitative rate at which $D(n)$ approaches 1, which yields a lower bound for $\delta(u)$ as $u \rightarrow 1^-$, given as (5.1) in Sec. 5.

We also compute a numerical example of a number n with $D(n) > 1/2$ and we consider some generalizations of the equation $ab = c$.

It is interesting to note that while there are product-free subsets with density arbitrarily close to 1, the density of *sum-free* subsets of finite abelian groups (written additively) is easily seen to be bounded by $1/2$ (see [4] for a complete characterization of the maximum density of sum-free subsets of various types of finite abelian groups).

2. THE MAIN THEOREM

In this section we show that there can be product-free sets of integers of density arbitrarily close to one, but not equal to one. Our main result is as follows.

Theorem 2.1. *There is a positive constant C and infinitely many integers n with*

$$D(n) > 1 - \frac{C}{(\log \log n)^{1 - \frac{1}{2}e \log 2}}.$$

Here the exponent $1 - \frac{1}{2}e \log 2 \approx 0.057915$.

Corollary 2.2. *For each real number $\epsilon > 0$ there is a positive integer n with $D(n) > 1 - \epsilon$.*

We first sketch the idea of the proof. Let $\Omega(m)$ denote the number of prime factors of m counted with multiplicity. Clearly for any fixed z , the set of numbers m with $z < \Omega(m) < 2z$ is product-free. Further, after Hardy and Ramanujan, we know that $\Omega(m)$ for numbers $m \leq x$ is usually concentrated near $\log \log x$. So if $z \approx \frac{2}{3} \log \log x$ (actually $\frac{e}{4}$ works out a little better than $\frac{2}{3}$), we have a product-free set that has the great preponderance of integers in $[1, x]$. With an extra device (see Lemma 2.3) for creating such a set that is periodic modulo some particular large number n , we obtain the result. The idea used bears some resemblance to that of Remark 2 and its proof in Hajdu, Schinzel, and Skalba [5].

Before giving the proof, we establish some preliminary lemmas. Let φ denote Euler's function and let $\text{rad}(n)$ denote the largest squarefree divisor of the positive integer n .

Lemma 2.3. *Suppose that n is a positive integer and \mathcal{D} is a product-free set of divisors of $n/\text{rad}(n)$. Then*

$$\mathcal{S}_{\mathcal{D}} := \{s \in \mathbb{Z}/n\mathbb{Z} : \gcd(s, n) \in \mathcal{D}\}$$

is product-free and

$$|\mathcal{S}_{\mathcal{D}}| = \varphi(n) \sum_{d \in \mathcal{D}} \frac{1}{d}.$$

Proof. Suppose $s_1, s_2 \in \mathcal{S}_{\mathcal{D}}$ with $\gcd(s_i, n) = d_i \in \mathcal{D}$ for $i = 1, 2$. We have $\gcd(s_1 s_2, n) = \gcd(d_1 d_2, n) = d_3$, say. If $d_3 \nmid n/\text{rad}(n)$, then by hypothesis $d_3 \notin \mathcal{D}$, so $s_1 s_2 \notin \mathcal{S}_{\mathcal{D}}$. On the other hand, if $d_3 \mid n/\text{rad}(n)$, then $d_3 = d_1 d_2$, so again by hypothesis, $d_3 \notin \mathcal{D}$ and $s_1 s_2 \notin \mathcal{S}_{\mathcal{D}}$. Thus, $\mathcal{S}_{\mathcal{D}}$ is product-free and it remains to compute its cardinality. For $d \in \mathcal{D}$, we have

$$\{s \in \mathbb{Z}/n\mathbb{Z} : \gcd(s, n) = d\} = \{jd : j \in \mathbb{Z}/(n/d)\mathbb{Z}, \gcd(j, n/d) = 1\}.$$

Thus, $|\mathcal{S}_{\mathcal{D}}| = \sum_{d \in \mathcal{D}} \varphi(n/d)$. But, by hypothesis, we have $\text{rad}(n/d) = \text{rad}(n)$ for $d \in \mathcal{D}$, so that $\varphi(n/d) = \varphi(n)/d$. This completes the proof. \square

For an integer $n > 1$, let $P(n)$ denote the largest prime factor of n and let $P(1) = 1$. As above, we let $\Omega(n)$ denote the number of prime factors of n , counted with multiplicity. We use the notation $f(x) \asymp g(x)$ if there are positive constants c_1, c_2 such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in some stated domain for the variable x . Lemma 2.4 and Corollary 2.5 below are standard results, cf. Exercises 04 and 05 in [6]; we give the details for completeness.

Lemma 2.4. *Uniformly for real numbers x, z with $x \geq 2$ and $0 < z < 2$,*

$$\sum_{P(n) \leq x} \frac{z^{\Omega(n)}}{n} \asymp \frac{1}{2-z} (\log x)^z.$$

Proof. We have

$$\begin{aligned} \sum_{P(n) \leq x} \frac{z^{\Omega(n)}}{n} &= \prod_{p \leq x} \left(1 + \frac{z}{p} + \frac{z^2}{p^2} + \cdots \right) = \prod_{p \leq x} \left(1 - \frac{z}{p} \right)^{-1} \\ &= \prod_{p \leq x} \left(1 - \frac{1}{p} \right)^{-z} \prod_{p \leq x} \left(1 - \frac{1}{p} \right)^z \left(1 - \frac{z}{p} \right)^{-1}. \end{aligned}$$

By the theorem of Mertens we have $\prod_{p \leq x} (1 - 1/p)^{-z} \sim e^{\gamma z} (\log x)^z$ uniformly for z in the interval $(0, 2)$, as $x \rightarrow \infty$, where γ is the Euler–Mascheroni constant. Thus, it suffices to prove that the second product above is of magnitude $1/(2 - z)$. Using the power series for $\log(1 - t)$, we have

$$\begin{aligned} \log \left(\prod_{p \leq x} \left(1 - \frac{1}{p}\right)^z \left(1 - \frac{z}{p}\right)^{-1} \right) &= \sum_{p \leq x} \left(z \log \left(1 - \frac{1}{p}\right) - \log \left(1 - \frac{z}{p}\right) \right) \\ &= z \log \frac{1}{2} - \log \left(1 - \frac{z}{2}\right) + O \left(\sum_{3 \leq p \leq x} \frac{1}{p^2} \right) = -\log(2 - z) + O(1). \end{aligned}$$

This then completes the proof of the lemma. \square

We will use the entropy-like function $Q(x)$ defined for $x > 0$ by

$$Q(x) = x \log x - x + 1.$$

Note that $Q(x) \geq 0$ for all $x > 0$ with equality only at $x = 1$.

Corollary 2.5. *Uniformly for real numbers α, β, x with $0 < \alpha \leq 1 \leq \beta < 2$ and $x \geq 3$, we have*

$$\sum_{\substack{P(n) \leq x \\ \Omega(n) \leq \alpha \log \log x}} \frac{1}{n} \ll (\log x)^{1-Q(\alpha)}, \quad \sum_{\substack{P(n) \leq x \\ \Omega(n) \geq \beta \log \log x}} \frac{1}{n} \ll \frac{1}{2 - \beta} (\log x)^{1-Q(\beta)}.$$

Proof. We have

$$\begin{aligned} \sum_{\substack{P(n) \leq x \\ \Omega(n) \leq \alpha \log \log x}} \frac{1}{n} &\leq \sum_{P(n) \leq x} \frac{\alpha^{\Omega(n) - \alpha \log \log x}}{n} \\ &= \sum_{P(n) \leq x} \frac{\alpha^{\Omega(n)}}{n} (\log x)^{-\alpha \log \alpha} \ll (\log x)^{\alpha - \alpha \log \alpha}, \end{aligned}$$

using $0 < \alpha \leq 1$ and Lemma 2.4 with $z = \alpha$. Similarly, Lemma 2.4 with $z = \beta$ gives

$$\sum_{\substack{P(n) \leq x \\ \Omega(n) \geq \beta \log \log x}} \frac{1}{n} \leq \sum_{P(n) \leq x} \frac{\beta^{\Omega(n) - \beta \log \log x}}{n} \ll \frac{1}{2 - \beta} (\log x)^{\beta - \beta \log \beta}.$$

This completes the proof of the corollary. \square

Proof of Theorem 2.1. Let x be a large real number, let ℓ_x denote the least common multiple of the integers in $[1, x]$, and let $n_x = \ell_x^2$. Thus,

by the prime number theorem, we have $n_x = e^{(2+o(1))x}$ as $x \rightarrow \infty$, so that

$$(2.1) \quad \log \log n_x = \log x + O(1).$$

Let

$$\mathcal{D}_x = \left\{ d \mid \ell_x : \frac{e}{4} \log \log x < \Omega(d) < \frac{e}{2} \log \log x \right\}.$$

We note that each $d \in \mathcal{D}_x$ divides $n_x/\text{rad}(n_x)$ and that \mathcal{D}_x is product-free. Thus, by Lemma 2.3 we find that

$$\mathcal{S}_{\mathcal{D}_x} := \{a \in \mathbb{Z}/n_x\mathbb{Z} : \gcd(a, n_x) \in \mathcal{D}_x\}$$

is a product-free subset of $\mathbb{Z}/n_x\mathbb{Z}$, with density $\mathcal{D}(\mathcal{S}) = \frac{\varphi(n_x)}{n_x} \sum_{d \in \mathcal{D}_x} \frac{1}{d}$. Using (2.1) it suffices to show that for some positive constant c and x sufficiently large,

$$(2.2) \quad \frac{\varphi(n_x)}{n_x} \sum_{d \in \mathcal{D}_x} \frac{1}{d} \geq 1 - \frac{c}{(\log x)^{1-\frac{1}{2}e \log 2}}.$$

We have

$$\sum_{d \in \mathcal{D}_x} \frac{1}{d} \geq \sum_{d \mid \ell_x} \frac{1}{d} - \sum_{\substack{P(d) \leq x \\ \Omega(d) \leq \frac{e}{4} \log \log x}} \frac{1}{d} - \sum_{\substack{P(d) \leq x \\ \Omega(d) \geq \frac{e}{2} \log \log x}} \frac{1}{d}.$$

Since $1 - Q(\frac{e}{4}) = 1 - Q(\frac{e}{2}) = \frac{1}{2}e \log 2$, Corollary 2.5 implies there is some absolute constant $c' > 0$ with

$$\sum_{d \in \mathcal{D}_x} \frac{1}{d} \geq \sum_{d \mid \ell_x} \frac{1}{d} - c'(\log x)^{\frac{1}{2}e \log 2}.$$

Now, letting σ denote the sum-of-divisors function,

$$\begin{aligned} \sum_{d \mid \ell_x} \frac{1}{d} &= \frac{\sigma(\ell_x)}{\ell_x} = \prod_{p^a \parallel \ell_x} \frac{p^{a+1} - 1}{p^a(p-1)} = \prod_{p \leq x} \frac{p}{p-1} \prod_{p^a \parallel \ell_x} \left(1 - \frac{1}{p^{a+1}}\right) \\ &\geq \prod_{p \leq x} \frac{p}{p-1} \cdot \left(1 - \frac{1}{x}\right)^{\pi(x)} \geq \prod_{p \leq x} \frac{p}{p-1} \cdot \left(1 - \frac{\pi(x)}{x}\right), \end{aligned}$$

where $\pi(x)$ denotes the prime-counting function. Thus, since $\varphi(n_x)/n_x = \prod_{p \leq x} (p-1)/p$,

$$\frac{\varphi(n_x)}{n_x} \sum_{d \in \mathcal{D}_x} \frac{1}{d} \geq 1 - \frac{\pi(x)}{x} - c'(\log x)^{\frac{1}{2}e \log 2} \prod_{p \leq x} \frac{p-1}{p}.$$

Using the theorem of Mertens for the product and the Chebyshev estimate $\pi(x) \ll x/\log x$, we obtain (2.2), completing the proof of Theorem 2.1. \square

Remark 2.6. It is possible to uniformly save a factor $\sqrt{\log \log x}$ in Corollary 2.5 under the strengthened hypothesis that $\alpha \in [\epsilon, 1 - \epsilon]$ and $\beta \in [1 + \epsilon, 2 - \epsilon]$, where $\epsilon > 0$ is fixed but arbitrary. This gives a slightly stronger version of Theorem 2.1: There is a positive constant C such that

$$(2.3) \quad D(n) > 1 - \frac{C}{(\log \log n)^{1 - \frac{1}{2}e \log 2} \sqrt{\log \log \log n}} \text{ for infinitely many } n.$$

The details are presented in a sequel paper [7], where the principal result is that (2.3), apart from the constant C , is best possible.

Remark 2.7. For a set \mathcal{S} of positive integers, let $\mathcal{S}(x) = \mathcal{S} \cap [1, x]$. If \mathcal{S} is product-free with least member a , then its upper asymptotic density, defined as

$$\bar{d}(\mathcal{S}) := \limsup_{x \rightarrow \infty} \frac{1}{x} |\mathcal{S}(x)|,$$

satisfies $\bar{d}(\mathcal{S}) \leq 1 - \frac{1}{2a}$. To see this, suppose $x \geq a$ is arbitrary. Since $\mathcal{S}(x) \setminus \mathcal{S}(x/a)$ lies in $(x/a, x]$, we have $|\mathcal{S}(x)| - |\mathcal{S}(x/a)| \leq x - \lfloor x/a \rfloor$. Also, multiplying each member of $\mathcal{S}(x/a)$ by a creates products in $[1, x]$ which cannot lie in \mathcal{S} , so we have $|\mathcal{S}(x)| \leq x - |\mathcal{S}(x/a)|$. Adding these two inequalities leads to $|\mathcal{S}(x)| \leq x - \frac{1}{2} \lfloor x/a \rfloor$, which proves the assertion.

3. GENERALIZATIONS

If k, j are positive integers, we say a set of integers (or residue classes in $\mathbb{Z}/n\mathbb{Z}$) is (k, j) -product-free if there is no solution to $a_1 a_2 \dots a_k = b_1 b_2 \dots b_j$ with all $k + j$ letters being elements of the set. If $k = j$ then only the empty set is (k, j) -product-free. Indeed, if a is an element of the set, the equation $a^k = a^k$ shows that we cannot avoid $a_1 a_2 \dots a_k = b_1 b_2 \dots b_j$. Thus we restrict to cases where $k \neq j$, and we may as well assume that $k > j$. The case of $k = 2, j = 1$ is the unadorned definition of product-free that was considered in the last section. In this section we record the following simple generalization.

Theorem 3.1. *For each real number $\epsilon > 0$ and integer $m \geq 3$ there is a positive integer n and a subset \mathcal{S} of $\mathbb{Z}/n\mathbb{Z}$ of cardinality at least $(1 - \epsilon)n$ that is simultaneously (k, j) -product-free for all positive integers $k > j$ with $k + j \leq m$.*

Proof. As in the proof of Theorem 2.1, let ℓ_x denote the least common multiple of the integers in $[1, x]$, but now we let $n_x = \ell_x^m$, and

$$\mathcal{D}_x = \left\{ d \mid \ell_x : \left(1 - \frac{1}{m}\right) \log \log x < \Omega(d) < \left(1 + \frac{1}{m}\right) \log \log x \right\}.$$

Let $k > j$ be positive integers with $k + j \leq m$. If $d_1, \dots, d_k \in \mathcal{D}_x$ and also $d'_1, \dots, d'_j \in \mathcal{D}_x$, it is easy to see that $d = d_1 \dots d_k$ and $d' = d'_1 \dots d'_j$ are divisors of n_x . In addition, $d \neq d'$, since $\Omega(d) > k(1 - \frac{1}{m}) \log \log x \geq j(1 + \frac{1}{m}) \log \log x > \Omega(d')$. Thus, \mathcal{D}_x is (k, j) -product-free as is the set $\mathcal{S}_{\mathcal{D}_x}$ (cf. Lemma 2.3). As in the proof of Theorem 2.1 it suffices to show that for each $\epsilon > 0$,

$$\frac{\varphi(n_x)}{n_x} \sum_{d \in \mathcal{D}_x} \frac{1}{d} \geq 1 - \epsilon$$

for all sufficiently large x depending on ϵ . Already from the proof of Theorem 2.1, we have

$$\frac{\varphi(n_x)}{n_x} \sum_{d \mid \ell_x} \frac{1}{d} \geq 1 - \frac{\pi(x)}{x} \sim 1$$

as $x \rightarrow \infty$. Since $\varphi(n_x)/n_x \sim 1/(e^\gamma \log x)$ as $x \rightarrow \infty$, it suffices to show that

$$(3.1) \quad \sum_{\substack{d \mid \ell_x \\ d \notin \mathcal{D}_x}} \frac{1}{d} = o(\log x) \quad \text{as } x \rightarrow \infty.$$

Letting $\delta_1 = Q(1 - 1/m)$ and $\delta_2 = Q(1 + 1/m)$, we have $\delta_1, \delta_2 > 0$. Using Corollary 2.5,

$$\sum_{\substack{d \mid \ell_x \\ \Omega(d) \leq (1 - \frac{1}{m}) \log \log x}} \frac{1}{d} \leq (\log x)^{1 - \delta_1/2}, \quad \sum_{\substack{d \mid \ell_x \\ \Omega(d) \geq (1 + \frac{1}{m}) \log \log x}} \frac{1}{d} \leq (\log x)^{1 - \delta_2/2}$$

for all large x . Thus, we have (3.1), which completes the proof of the theorem. \square

Returning to the case when $k = j$, we can redefine the notion of (k, k) -product-free to mean that the equation $a_1 a_2 \dots a_k = b_1 b_2 \dots b_k$ implies that $\{a_1, a_2, \dots, a_k\} = \{b_1, b_2, \dots, b_k\}$ as multisets. For example, the primes are (k, k) -product-free for every k . This is essentially a best-possible result, for as shown by Erdős [3] in 1938, if \mathcal{S} is a subset of the positive integers which is $(2, 2)$ -product-free, then the number of members of \mathcal{S} in $[1, x]$ is $\pi(x) + O(x^{3/4})$.

The equation $abc = d^2$ was recently considered in [5], where it was shown (see Corollary 1) that if \mathcal{S} is a set of integers such that

$$abc = d^2 \quad \text{has no solution with } a, b, c \in \mathcal{S}, \quad d \text{ arbitrary,}$$

then the lower asymptotic density of \mathcal{S} is at most $1/2$. This result was inadvertently misquoted in [9], where it was asserted that such a result holds with all of $a, b, c, d \in \mathcal{S}$. In fact, this is false since Theorem 3.1 applied with $(k, j) = (3, 2)$ implies the complementary result that for any $\epsilon > 0$ there exists a set \mathcal{S} of density exceeding $1 - \epsilon$ such that

$$(3.2) \quad abc = d^2 \quad \text{has no solution with } a, b, c, d \in \mathcal{S}.$$

More precisely, it gives:

Corollary 3.2. *For each real number $\epsilon > 0$, there is a positive integer n and a subset \mathcal{S} of $\mathbb{Z}/n\mathbb{Z}$ of cardinality at least $(1 - \epsilon)n$ such that $abc = d^2$ has no solution with $a, b, c, d \in \mathcal{S}$.*

4. A NUMERICAL EXAMPLE

In this section we give the details for a number N for which there exists a product-free subset of $\mathbb{Z}/N\mathbb{Z}$ of size larger than $N/2$. Our example is very large; it would be of interest to see if a substantially smaller number could be found.

Let \mathcal{P} denote the set of the first 10,000,000 primes and let Q be their product. For each positive integer j , let

$$\sigma_j = \sum_{p \in \mathcal{P}} \frac{1}{p^j}, \quad S_j = \sum_{\substack{\text{rad}(m) | Q \\ \Omega(m) = j}} \frac{1}{m}.$$

We have computed these sums for j up to 13, finding that to 6 decimal places,

$$\begin{aligned} \sigma_1 &= 3.206219, & \sigma_2 &= 0.452247, & \sigma_3 &= 0.174763, & \sigma_4 &= 0.076993, \\ \sigma_5 &= 0.035755, & \sigma_6 &= 0.017070, & \sigma_7 &= 0.008284, & \sigma_8 &= 0.004061, \\ \sigma_9 &= 0.002004, & \sigma_{10} &= 0.000994, & \sigma_{11} &= 0.000494, & \sigma_{12} &= 0.000246, \\ \sigma_{13} &= 0.000123 \end{aligned}$$

and

$$\begin{aligned} S_1 &= 3.206219, & S_2 &= 5.366043, & S_3 &= 6.276492, & S_4 &= 5.796977, \\ S_5 &= 4.529060, & S_6 &= 3.130763, & S_7 &= 1.976769, & S_8 &= 1.167289, \\ S_9 &= 0.656256, & S_{10} &= 0.356061, & S_{11} &= 0.188345, & S_{12} &= 0.097866, \\ S_{13} &= 0.050226. \end{aligned}$$

Concerning these calculations, we note that the computation for $\sigma_1 = S_1$ is the most time consuming. The other values of σ_j represent the starts of rapidly converging series, and in fact these values can be found

on the web as values of the “prime zeta function.” The remaining values of S_j are easily computed by a hand calculator using the identity

$$S_k = \frac{1}{k} \sum_{j=1}^k \sigma_j S_{k-j},$$

where by convention we take $S_0 = 1$ (see [8, page 23, (2.11)]).

Let

$$N = Q^{14} = \prod_{p \in \mathcal{P}} p^{14}$$

and let

$$\mathcal{D} = \{d \mid N : 3 \leq \Omega(d) \leq 5 \text{ or } 11 \leq \Omega(d) \leq 13\}.$$

A moment’s reflection shows that \mathcal{D} is product-free and that each member of \mathcal{D} divides $N/\text{rad}(N)$, and so from Lemma 2.3,

$$\mathcal{S}_{\mathcal{D}} = \{m \bmod N : \gcd(m, N) \in \mathcal{D}\}$$

is also product-free. Further,

$$(4.1) \quad \frac{|\mathcal{S}_{\mathcal{D}}|}{N} = \frac{\varphi(N)}{N} \sum_{d \in \mathcal{D}} \frac{1}{d}.$$

We may compute $\varphi(N)/N$ using σ_1 and σ_2 as follows:

$$\log \frac{\varphi(N)}{N} = \sum_{p \in \mathcal{P}} \log \left(1 - \frac{1}{p}\right) = -\sigma_1 - \frac{1}{2}\sigma_2 + \sum_{p \in \mathcal{P}} \left(\frac{1}{p} + \frac{1}{2p^2} + \log \left(1 - \frac{1}{p}\right)\right).$$

The remaining sum above is the start of a rapidly converging series, so we easily find that

$$(4.2) \quad \frac{\varphi(N)}{N} > 0.029542.$$

The sum in (4.1) is

$$\sum_{d \in \mathcal{D}} \frac{1}{d} = S_3 + S_4 + S_5 + S_{11} + S_{12} + S_{13} = 16.938967.$$

Thus, with (4.1) and (4.2), we have

$$\frac{|\mathcal{S}_{\mathcal{D}}|}{N} > (0.029542)(16.9389) > 0.5004.$$

This number N is very large, it is about $10^{1.09 \times 10^9}$. However, it is possible to reduce the exponents somewhat for the larger primes in N . Let N' be N divided by the 12th power of each prime dividing N that is above 10^6 . Then $D(N') > 0.5003N'$ and N' is about $10^{1.61 \times 10^8}$. We

have made some effort at finding a smaller example, say below 10^{10^8} , but we were not successful.

5. DENSITIES AND FURTHER PROBLEMS

Let $u \in [0, 1)$ be a real number and, as in the introduction, let $\mathcal{N}(u)$ denote the set of natural numbers n with $D(n) > u$. Since $D(mn) \geq D(n)$, it follows that if $n \in \mathcal{N}(u)$, so too is every multiple of n . Consequently $\mathcal{N}(u)$ has a logarithmic density, see [1, 2], denote this by $\delta(u)$. We have by Corollary 2.2 that $\delta(u) > 0$ for all $u \in [0, 1)$. We can say a bit more.

Proposition 5.1. *We have $\liminf_{n \rightarrow \infty} D(n) = 1/2$. Consequently for $0 \leq u < \frac{1}{2}$ the set $\mathcal{N}(u)$ has both a logarithmic density $\delta(u)$ and a natural density $d(u)$ satisfying $d(u) = \delta(u) = 1$.*

Proof. Let p be an odd prime and let a be a positive integer. The set of nonzero residues mod p^a which are the product of a power of p and a quadratic nonresidue mod p is product-free, and this shows that $D(p^a) \rightarrow \frac{1}{2}$ as $a \rightarrow \infty$ (recall that $D(n) < 1/2$ if $n/\text{rad}(n)$ does not have at least 6 distinct prime factors). In addition, the set of nonzero residues mod 2^a which are the product of a power of 2 and an integer that is 3 mod 4 is product-free, so that $D(2^a) \rightarrow \frac{1}{2}$ as $a \rightarrow \infty$. Since $D(p) \rightarrow \frac{1}{2}$ as $p \rightarrow \infty$ through the primes, it follows that $D(q) \rightarrow \frac{1}{2}$ as $q \rightarrow \infty$ through the prime powers (which include the primes). Hence for each real number $\epsilon > 0$, there are at most finitely many prime powers q with $D(q) \leq \frac{1}{2} - \epsilon$. Thus, if $D(n) \leq \frac{1}{2} - \epsilon$, it follows that each prime power dividing n must come from this set, forcing the set of such n to be finite as well. This proves the first statement in the proposition. Let $u \in [0, 1/2)$. By what we just proved, the set $\mathcal{N}(u)$ consists of all but finitely many natural numbers. This establishes the second statement in the proposition. \square

It follows from the principal results of [9] that $\delta(1/2) \leq 1.56 \times 10^{-8}$, and so with Proposition 5.1 it follows that $\delta(u)$ is not continuous in the variable u at $1/2$. From the numerical example in the last section, we have $\delta(1/2) > 10^{-1.62 \times 10^8}$. There is of course an enormous (multiplicative) gap between these two bounds for $\delta(1/2)$.

More generally Theorem 2.1 yields a lower bound for $\delta(u)$ as $u \rightarrow 1^-$. Setting $\alpha_0 := (1 - \frac{1}{2}e \log 2)^{-1} \approx 17.26659$, we have

$$(5.1) \quad \delta(u) > 1 / \exp \exp ((C / (1 - u))^{\alpha_0}).$$

Note that (2.3) allows a slight improvement in this estimate.

It seems likely that for each u , the set $\mathcal{N}(u)$ has an asymptotic density $d(\mathcal{N}(u))$. General facts about asymptotic densities give $\underline{d}(\mathcal{N}(u)) \leq \delta(u) \leq \overline{d}(\mathcal{N}(u))$, and a natural density $d(u) = \delta(u)$ exists for those values with $\underline{d}(\mathcal{N}(u)) = \overline{d}(\mathcal{N}(u))$. Our proofs show that $\underline{d}(\mathcal{N}(u)) > 0$ for $0 < u < 1$ and $\overline{d}(\mathcal{N}(u)) < 1$ for $u \geq \frac{1}{2}$.

As asked in [9], is it true that for $u \geq 1/2$, the “primitive” members of $\mathcal{N}(u)$ (namely, they are not divisible by any other member of $\mathcal{N}(u)$) are all squarefull? If so, then it would follow that the asymptotic density of $\mathcal{N}(u)$ exists for each value of u .

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